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Optimal Fuzzy Control of Inverted Pendulum on Cart

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In this paper, Takagi Sugeno Kang (TSK) fuzzy modeling and Exact Fuzzy modeling methodology for nonlinear MIMO dynamic systems are used to model the bench mark nonlinear system, Inverted pendulum on cart and cart drive dynamics. The cart drive dynamics is modeled as a nonlinear second order system with input saturation and nonlinear spring. Nonlinear optimal control methodology based on TSK fuzzy model is used to design the optimal controller for above system. The controlled system stability is ensured by the Fuzzy optimal control theory. The simulation results are compared with a linear optimal controller and the performance improvement demonstrated.

NOMENCLATURE

X	Sate vector
A	System matrix
B	Input matrix
M	Membership function
h	weighting coefficient
F	Feedback matrix
K	Solution of matrix Ricatti equation
Q	State weighting matrix
R	Control input weighting matrix
θ	Angular position of pendulum
f_i	Nonlinear scalar functions
β_i	Upper bound of f_i
α_i	Lower bound of f_i

I. INTRODUCTION

There are several approaches for the control of a nonlinear system. A typical and simple approach is the feedback stabilization of nonlinear systems where, a linear feedback control is designed for the linearization of the system about a nominal operating point. This approach however only renders a local result in general. Other approaches such as feedback linearization would normally require a considerable degree of sophistication and tend to result in rather complicated controllers.

The fuzzy controllers, in general is a model free approach for controlling nonlinear and uncertain plants. The first generation (Mamdani) fuzzy modeling and control approach belongs to the above class. The first generation fuzzy control mainly focused on translating knowledge expressed in linguistic form to a control algorithm which is practically implimentable. Many real life systems implemented the Mamdani type fuzzy control and found to have very good performance and robustness. But a control system theoretic approach to analyse the stability and optimality issues in the Mamdani type fuzzy control has been found difficult and impossible. Formal analysis method requires a model based approach. TSK fuzzy model is a second generation fuzzy modeling approach which uses fuzzy blending of linear models to represent any nonlinear plant. This approach has enabled system theoretic aspects such as stability and optimality analysis possible.

In TSK fuzzy[4] modeling the nonlinear plant is first approximated by fuzzy blending of different linear models in different state space region. In TSK model based control design, for each local linear model a linear feedback controller

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is designed and the overall controller is synthesized as a fuzzy blending of local linear controllers. The overall controller is nonlinear. This approach of fuzzy controller design is called Parallel Distributed Compensation (PDC)[6] developed by Wan, Tanaka and Griffin. The stability analysis for PDC is well developed. Optimal controller design based on TSK fuzzy model was developed by Wu and Lin [3], using principles of dynamic programming. In all these cases the key is the accuracy of the fuzzy model obtained for the true system on which stability and optimality results rests. Exact fuzzy representation of a known scalar function was derived by Watkin and Kosko[1,8]. The exact fuzzy modeling for nonlinear MIMO dynamic system was developed by Mohanlal and Kaimal[2]. In exact fuzzy modeling instead of local linear models, boundary linear models are used for fuzzy blending to exactly represent a class of nonlinear systems. All the stability analysis and optimal control results are directly applicable to the exact fuzzy model based fuzzy controller design.

TSK fuzzy modeling and exact fuzzy modeling of nonlinear dynamic system is given in section II. Stable and optimal controller design is described in section III. Optimal controller design for Pendulum on cart system with nonlinear cart drive dynamics is developed in section IV.

II. TSK Fuzzy Modeling

In TSK fuzzy modeling [4] nonlinear system is approximated as a convex sum of local linear systems. In general this fuzzy modeling for a system represented by state equations, with n states, is of the form Rule i: If $x_1(t)$ is M_{i1}, $x_2(t)$ is M_{i2} ...... $x_n(t)$ is M_{in} then $\dot{X}(t) = A_i X(t) + B_i u(t)$, for i = 1,2.....r, where 'r' represent the total number of subsystem considered (that is the total number of if-then rules) and M_{ij} is a fuzzy set.

The overall fuzzy model is achieved by fuzzy blending of these linear system models. For a given X, u the final fuzzy system is inferred as

$$\dot{X}(t) = \frac{\sum_{i=1}^{r} w_i(X)\{A_i X(t) + B_i u(t)\}}{\sum_{i=1}^{r} w_i(X)} \tag{2.1}$$

where $w_i(X) = \prod_{j=1}^{n} M_{ij}(x_j) \geq 0$, $i=1....r$ is called the weight. Each weight $w_i(X)$ is a function of X, which makes the blended system nonlinear. $M_{ij}(x_j)$ is the grade of membership of $x_j(t)$ in M_{ij}. Each linear component $A_i X(t) + B_i u(t)$ is called a subsystem. Eqn (2.1) can be rewritten as

$$\dot{X}(t) = \sum_{i=1}^{r} h_i(x)[A_i X(t) + B_i u(t)]$$

$$\sum_{i=1}^{r} h_i(x) = 1, h_i(x) \geq 0 \tag{2.2}$$

From Eqn. (2.2) it is evident that the overall system is convex sum of r subsystems. The convexity property is the key for stability analysis and fuzzy optimal control results.

A. Exact Fuzzy Modeling

General TSK fuzzy modeling results in an approximate model for the nonlinear system, even in situations where the nonlinear equations are known apriori. If the nonlinear equations are known apriori, we can utilize blending of boundary linear models [2] for exact representation of nonlinear systems. This exact fuzzy modeling is an extension of the exact fuzzy representation of a scalar function [1,8].

According to Watkins's: A model needs just two rules to represent any bounded non constant scalar function $f: R \rightarrow R$. This represents f in to two parts A_1 and A_2 (A_2 is the complement of A_1 part).

Theorem: A SAM $F: R \rightarrow R$ with two rules of the form

"If X = A_1 then Y = B_1" and "if X= A_2 then Y= B_2" can represent a bounded non constant function $f: R \rightarrow R$ in the sence that F(x) = f(x) for all 'x belonging to R.'

If we have full knowledge of the function, that is the function itself and its bounds (Boundedness: lets us define the lower and upper bounds as α and β respectively) then define the set function of then part set A_1 as

$$M_1(x) = \frac{\beta - f(x)}{\beta - \alpha} \tag{2.3}$$

The set function of then part set A_2 as

$$M_2(x) = 1 - M_1(x) \tag{2.4}$$

We can get back the function $f(x)$ by

$$f(x) = \frac{M_1(x)\alpha + M_2(x)\beta}{M_1 + M_2} \tag{2.5}$$

B. Extension to MIMO Systems

The result of section II-A was extended to dynamic systems [2]. Nonlinear dynamic system of the form $\dot{X} = A(X)X + B(X)U$ where $X \in R^n$ is the state vector, $U \in R^m$ is the input vector, $m < n$, and $A(X) \in R^{n \times n}$ and $B(X) \in R^{n \times m}$ and elements of $A(X)$ and $B(X)$ are scalar valued functions of the state vector: then there exist convex weighing functions h_i such that the system can be represented as $\dot{X} = \sum_{i=1}^{p} h_i[A_i X + B_i U]$ where $p \leq n^2 + nm$ and $\sum_{i=1}^{p} h_i = 1$

and $0 \le hi \le 1$ for all i = 1,2...p.(A_i, B_i) are linear subsystems constructed using the bounds of the scalar valued functions in $A(X)$ and $B(X)$.

III. Stable and Optimal Controller Design

Stability of TSK fuzzy model for continuous and discrete time has been developed by Tanaka and Sugeno[7] based on linear matrix inequalities (LMI). This is an extension of classical Lyapunov inequality in linear controller design. The PDC based fuzzy control design and the stability analysis was developed based on LMI. Optimal control result for TSK was developed by Wu and Lin [3] based on dynamic programming. We utilize these results for the fuzzy controller design for inverted pendulum on cart, for which exact fuzzy model have been developed in this paper.

A. Parallel Distributed Compensation

The idea is to design a compensator for each rule of the fuzzy model. The fuzzy controller shares the same fuzzy sets with the system. For each rule linear control design techniques can be used. Here for each rule the controller designed is a linear feedback controller. The resulting overall controller is a fuzzy blending of each individual controller and it is nonlinear in general even though the individual controller is linear.

Consider the system with rule i: If $x_1(t)$ is M_{i1}, $x_2(t)$ is M_{i2} ...... $x_n(t)$ is M_{in}. Then the controller for this rule is

$$u_i(t) = -F_i X(t) \tag{3.1}$$

The overall controller is a fuzzy blending of these individual controllers, hence the overall controller output is

$$u(t) = \frac{-\sum_{i=1}^{r} w_i(X)\, F_i\, X(t)}{\sum_{i=1}^{r} w_i(X)} \tag{3.2}$$

$$u(t) = -\sum_{i=1}^{n} h_i(X)\, F_i\, X(t) \tag{3.3}$$

From (2.1) and (3.3) the controlled system can be represented as

$$\dot{X}(t) = \frac{-\sum_{i=1}^{r}\sum_{j=1}^{r} w_i(X)\, w_j(X)(A_i - B_i F_j)\, X(t)}{\sum_{i=1}^{r}\sum_{j=1}^{r} w_i(X)\, w_j(X)} \tag{3.4}$$

This is a convex sum of fuzzy systems $\phi_{ij} = (A_i - B_i F_j)$. For continuous time systems the overall system will be stable if there exist a common positive definite matrix P such that $\phi_{ij}^T P + P^T \phi_{ij} < 0, i, j = 1,2,...r$. Hence if all these subsystems are stable the overall system is stable. For discrete time systems the stability of the sub systems does not ensure stability of the overall system. The controlled closed loop discrete time system is stable if there exists a positive definite matrix P such that $\psi_{ij}^T P \psi_{ij} - P < 0, i, j = 1,2,...r$, where ψ_{ij} is the system matrix. In general r^2 inequalities are to be satisfied for the overall system to be stable.

According to optimality results from local concept approach [3] the overall optimal controller is a convex sum of r optimal feedback controllers instead of r^2 systems in (3.4). Hence we can write the overall controlled system as

$$\dot{X}(t) = \sum_{i=1}^{r} h_i(X)(A_i - B_i F_i)\, X(t) \tag{3.5}$$

Thus with local concept approach the stability analysis of overall controlled system is simplified. The stability of the controlled system is derived in [3].

B. Optimal Controller Based on TSK Fuzzy Model

For the fuzzy system in (2.1) and controller in (3.5), if all A_i, B_i matrix is completely state controllable and all states are observable and if Q and R are Positive Definite weighing matrices for the state and input and if all subsystems are time invariant then there exists a unique symmetric positive semi-definite solution $\pi_\infty^{\ i}$, of steady state Ricatti equation:

$$A_i^T K + K A_i - K B_i R^{-1} B_i^T K + Q = 0 \tag{3.5a}$$

The asymptotically local optimal fuzzy control law is

$$u_i^*(t) = -B_i^T R^{-1} \pi_\infty^{\ i} X^*(t) \text{ , i = 1,2...r}$$

$$u_i^*(t) = -F_i X^*(t) \tag{3.6}$$

$X^*(t)$ indicates the optimal trajectory. The i[th] optimal local feedback subsystem is, $\dot{X}^*(t) = (A_i - B_i F_i) X^*(t)$ asymptotically and exponentially stable.

For the controller in (3.6) if A_i, B_i are completely controllable and all states are observable then the overall fuzzy system described by $\dot{X}^*(t) = \sum_{i=1}^{r} h_i(X^*)(A_i - B_i F_i) X^*(t)$ is exponentially stable. In this work the optimal fuzzy controller will be called as Fuzzy Quadratic Regulator (FQR).

It is important to note that the controlled system performance depends on the accuracy of the fuzzy model based on which the optimal design is carried out. Since in this work optimal control design is based on exact fuzzy model, the performance will be better.

IV. Example of Optimal Fuzzy Controller

A. Nonlinear system model for pendulum on cart

The equations of motion of Pendulum on cart system is

$$\ddot{\theta} = \frac{g \sin\theta - aml\dot{\theta}^2 \sin\theta\cos\theta - au\cos\theta}{\frac{4}{3}l - aml\cos^2\theta}$$

$$\ddot{x} = a\left\{u + ml\dot{\theta}^2 \sin\theta - ml\ddot{\theta}\cos\theta\right\} \tag{4.1}$$

Where, g: acceleration due to gravity, $a = \frac{1}{M+m}$, M: mass of cart, m: mass of pendulum, 2l : length of pendulum, u: control force. $x, \dot{x}, \ddot{x}$: Position, velocity, acceleration of the cart respectively. $\theta, \dot{\theta}, \ddot{\theta}$: Angular position, angular velocity, angular acceleration of the pendulum respectively. Linear Cart dynamics is $\ddot{u} + 19\pi\dot{u} + 36\pi^2 u = 36\pi^2 \times 15v$, Here v is the control voltage. The cart system natural frequency ω_n is assumed to be 6Π and the steady state gain of the system is taken as 15.

In this work the nonlinearities of cart dynamics is also considered. The main nonlinearities to be included are input saturation and nonlinear spring. Hence the nonlinear cart dynamics can be expressed as

$$\ddot{u} + 19\pi\dot{u} - f_8(u)u = f_7(v, vsat)v \tag{4.2}$$

Where the function $f_7(.)$ represents the effect of input saturation and the function $f_8(.)$ shows nonlinear spring effect.

Input saturation accounts to the inefficiency of the system to respond to an input value above a specified limit. We had modeled input saturation as a reduction in function $f_7(.)$ as the input is above the saturation limit.

if (v > vsat)

$$f_7 = \frac{36\pi^2 \times 15 \times v_{sat}}{|v|} \tag{4.3}$$

else

$$f_7 = 36\pi^2 \times 15 \tag{4.3a}$$

The function $f_8(.)$ is modeled in two ways in this paper. Case1: As an inverted bell function which shows an increase in stiffness as the displacement increases above a limit. Here the representation is as follows

$$f_8(u) = -\left[\left(\frac{36\pi^2}{\left(1+\left(\frac{|u|}{a}\right)^{-2b}\right)}\right) + 36\pi^2\right] \tag{4.4}$$

where 'a' and 'b' control the slope and spread of the bell curve. The plot for this function is depicted in figure 2.

Case2: $f_8(u)$ is modeled as follows

if (u > U)

$$f_8(u) = -2 \times 36\pi^2 \text{ else} \tag{4.5}$$

$$f_8(u) = -36\pi^2$$

where U is the limiting value of u above which the spring is assumed to be nonlinear.

B. Selection of Auxiliary Functions

The modeling in this work rests up on selecting proper boundary limited functions, ie auxiliary scalar functions has to be identified.

The equations of motion of pendulum on cart can be arranged in such a way that they can be rewritten in terms of following scalar valued auxiliary functions $f_1(\theta), f_2(\theta, \dot{\theta}), f_3(\theta), f_4(\theta), f_5(\theta, \dot{\theta}), f_6(\theta), f_7(v, vsat), f_8(u)$, as follows

$$\ddot{\theta} = f_1(\theta)\theta + f_2(\theta, \dot{\theta})\dot{\theta} + f_3(\theta)u$$

$$\ddot{x} = f_4(\theta)\theta + f_5(\theta, \dot{\theta})\dot{\theta} + f_6(\theta)u$$

$$\ddot{u} = f_7(v, vsat)v - 19\pi\dot{u} + f_8(u)u$$

For the simulation purpose the values taken are M = 8kg, m = 2kg, 2l = 1m, g = 9.81m/s^2 then the auxiliary functions become

$$f_1(\theta) = \frac{3g\sin\theta}{\theta(2-.3\cos^2\theta)}$$

$$f_2(\theta, \dot{\theta}) = \frac{-0.3\dot{\theta}\sin\theta\cos\theta}{(2-.3\cos^2\theta)}$$

$$f_3(\theta) = \frac{-0.3\cos\theta}{(2-.3\cos^2\theta)}$$

$$f_4(\theta) = \frac{-0.3g\sin\theta\cos\theta}{\theta(2-.3\cos^2\theta)}$$

The functions $f_7(v, vsat)$ and $f_8(u)$ are as shown in equations. (4.3) and (4.4) or (4.3) and (4.5). The total system can be represented in state space with six state variables: $\theta, \dot{\theta}, x, \dot{x}, u, \dot{u}$ and input v. The general form is $\dot{X} = A(X)X + B(X)v$ with X, A, B as follows

$$X = [\theta, \dot{\theta}, x, \dot{x}, u, \dot{u}]^T$$

$$A(.) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ f_1 & f_2 & 0 & 0 & f_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ f_4 & f_5 & 0 & 0 & f_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & f_8 & -19\pi \end{bmatrix}$$

$$B(.) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & f_7 \end{bmatrix}$$

C. Premise Variables for Exact Fuzzy Modeling

In general for TSK model states are used as premise variables for modeling, in this work auxiliary functions are used as premise variables. Each of the scalar auxiliary functions requires 2 membership functions to represent it exactly in the defined domain. Hence there will be 16 membership functions and they are represented as

$$M_{j1} = \frac{\beta_j - f_j(x)}{\beta_j - \alpha_j}, M_{j2} = 1 - M_{j1}, j = 1,...8$$

Where $\beta_j = \max(f_j)$ and $\alpha_j = \min(f_j)$

Since there are 8 auxiliary functions and each has two membership values we need $2^8 = 256$ rules to exactly represent the system. The 256 rules are formed in such a way that

In i[th] rule, if function f_j is M_{j1} then f_j is replaced with α_j in A_i and B_i and if f_j is M_{j2} then f_j is replaced with β_j in A_i and B_i. The corresponding convex weighing function will be the product of corresponding membership function values. For example the rule 1 can be

Rule 1: If $f_1(\ )$ is M_{11} and $f_2(\ )$ is M_{21} and $f_3(\ )$ is M_{31} and $f_4(\ )$ is M_{41} and $f_5(\ )$ is M_{51} and $f_6(\ )$ is M_{61} and $f_7(\ )$ is M_{71} and $f_8(\ )$ is M_{81} then the system is

$\dot{X} = A_1(X)X + B_1(X)$ where,

$$A_1(.) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \alpha_1 & \alpha_2 & 0 & 0 & \alpha_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha_4 & \alpha_5 & 0 & 0 & \alpha_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \alpha_8 & -19\pi \end{bmatrix}$$

$$B_1(.) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \alpha_7 \end{bmatrix}$$

The corresponding convex weighing coefficient is

$$h_1(X) = \prod_{i=1}^{8} M_{i1}(X)$$

In a similar manner we can represent the 256 subsystems and their weighing functions.

D. Controller Design

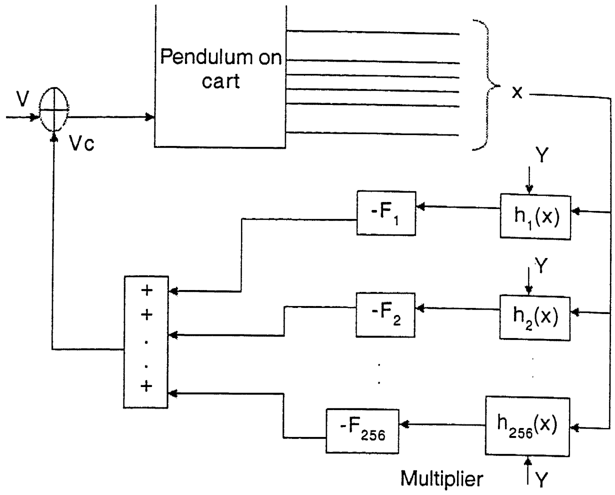
For each subsystem (1-256) the controller is designed with the optimal control result explained in section III-B. For each subsystem model the controller gain F_i is computed. The optimal global feedback fuzzy system is

$$\dot{X}^*(t) = \sum_{i=1}^{256} h_i(X^*)(A_i - B_i F_i) X^*(t)$$

The Q and R values selected for design are

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R = 1$$

This optimal global feedback fuzzy system is optimal in the sence that it is based on exact fuzzy model of the system considered in the domain of interest.



$$Y = [\theta, \dot{\theta}, u, v, vsat]$$

Fig 1: Pendulum on cart with Fuzzy Quadratic Regulator. F_i's represents each controller gain and $h_i(X)$'s the weighing functions. At each time instant various $h_i(X)$'s are computed and the controller output is computed as $V_c = -\sum_{i=1}^{256} h_i(X) F_i X(t)$

V. Simulation Results

The optimal controller designed in sec IV is used to control the pendulum on cart system in the restricted domain $-0.5rad \le \theta \le .5rad$, $-1.5rad/s \le \dot{\theta} \le 1.5rad/s$, $-12 \le u \le 12$ and $Vsat = 10$

The simulation results for desired pendulum angle $\theta = 0$ and cart position X = 1m are given.

Fig 2 shows the approximated nonlinear stiffness function of eqn. (4.4). Fig 3 and 4 shows the pendulum angle and Cart position respectively in this case for both LQR and FQR. Fig 5 and 6 shows the Cart position and pendulum angle respectively for both LQR and FQR, for the case where the nonlinear spring assumes the function expressed in eqn. (4.5).

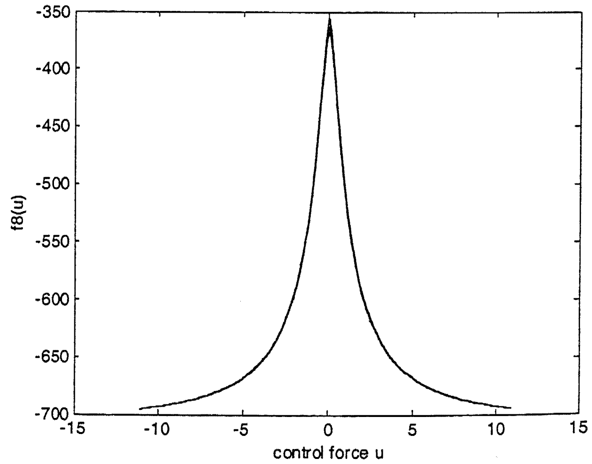


Figure 2. Bell shape function approximation of nonlinear spring

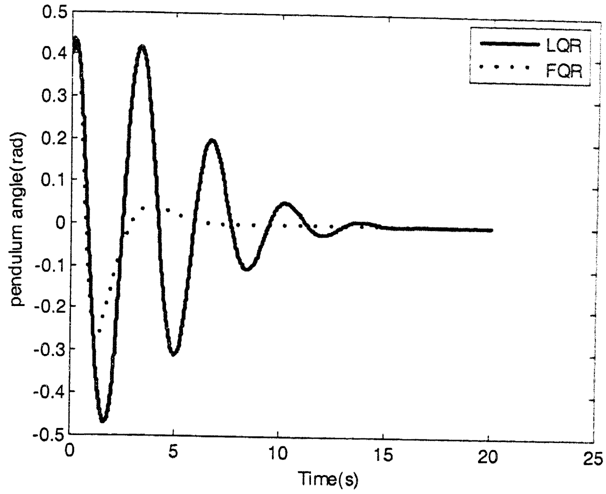


Figure 3. Pendulum Angle versus Time

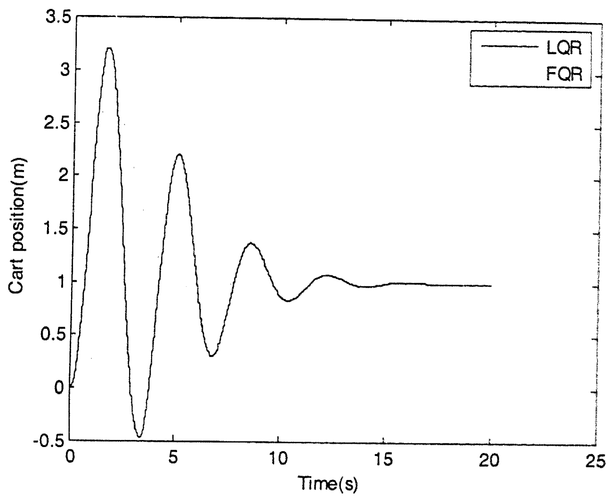


Figure 4. Cart Position versus Time

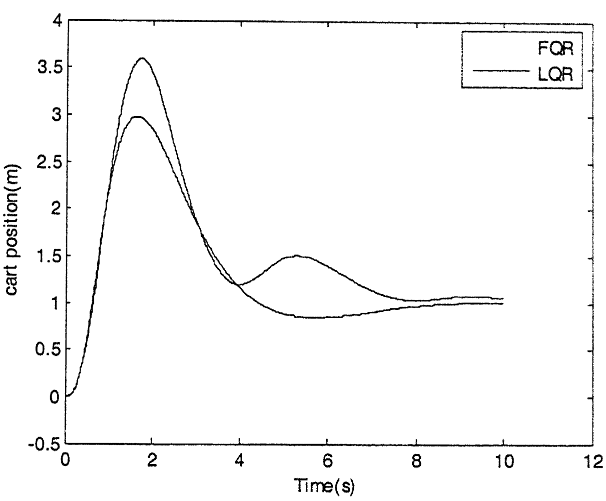


Figure 5. Cart Positions versus Time

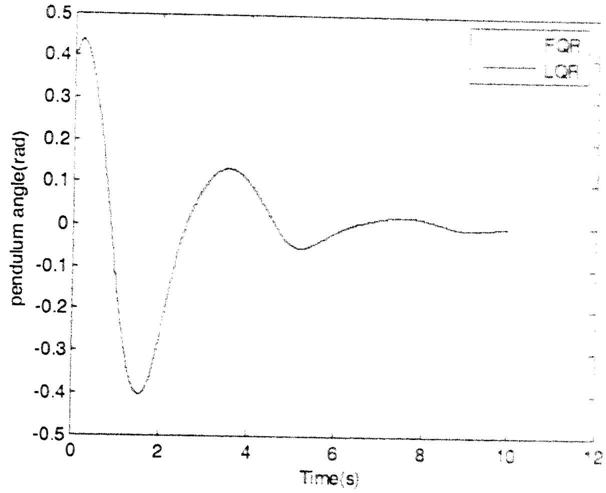


Figure 6. Pendulum Angle versus Time

The performance improvement in terms of reduction in peak overshoot, undershoot and settling time for FQR is evident from the above results.

VI. Conclusions

Optimal fuzzy controller (FQR) is designed for inverted pendulum on cart based on exact fuzzy TSK model. The Cart dynamics is also modeled as nonlinear. The results of FQR are compared with that of LQR. The performance improvement confirms the fuzzy optimal control methodology based on exact fuzzy model.

References

[1] Bart Kosko, Fuzzy engineering, Prentice Hall, 1997.

[2] Mohanlal P.P and Kaimal M.R., "Exact fuzzy model and optimal control of inverted pendulum on cart", IEEE conference on Decision and control, Dec 2002

[3] Shinq-Jen Wu and Chin-Teng Lin, "Optimal fuzzy Controller Design: Local concept approach ", IEEE Transactions on fuzzy systems, Vol.8, April 2000.

[4] T.Takagi and M.Sugeno, "Fuzzy identification of systems and its application to modeling and control", IEEE transactions on system, Man and Cybernetics,Vol 15 1995.

[5] H.O Wang, K. Tanaka and M. F. Griffin, "An approach to Fuzzy Control of nonlinear systems: stability and design issues", IEEE Transactions on fuzzy system, Vol.4, February 1996.

[6] H.O Wang, K. Tanaka and M. F. Griffin, "Parallel distributed compensation of nonlinear systems by TSK model", IEEE Transactions on fuzzy system,Dec1995

[7] K.Tanaka and M.Sugeno, "Stability analysis and design of fuzzy control systems", IEEE Transactions on Fuzzy sets and systems. Vol 45, Feb 1992.

[8] Watkins F.A., "The representation problem for additive fuzzy systems", Proceedings, IEEE fuzzy systems, Vol 1,March 1995